Homework 9

#1 Consider the following argument. Let X be a space and $A \subseteq X$. The sequence

$$0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$$

of singular chains is exact and $C_n(X,A)$ is free. Therefore

$$C_n(X) \approx C_n(A) \oplus C_n(X,A).$$

Hence $H_n(X) \approx H_n(A) \oplus H_n(X,A)$.

(a) Show by example that the conclusion above is false.

(b) Find the false assumption and say why it is false.

(c) Give a ~~additional~~ [illegible] condition for the conclusion to hold (other than A is contractible)

#2 Let X and Y be CW-complexes. Prove

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

See Massey, top of p. 230 for the product of CW-complexes.

#3 Let X and Y be CW-complexes and $f: X \to Y$ a cellular map (Massey, p. 232) f induces $f_r: H_r(X^r, X^{r-1}) \to H_r(Y^r, Y^{r-1})$, that is, a chain map $f_r: \mathcal{C}_r(X) \to \mathcal{C}_r(Y)$. and hence $f_{r*}: \mathcal{H}_r(X) \to \mathcal{H}_r(Y)$. Prove that the following diagram is commutative

$$\begin{array}{ccc} H_r(X) & \xrightarrow{\theta_X} & \mathcal{H}_r(X) \\ \downarrow f_* & & \downarrow f_{r*} \\ H_r(Y) & \xrightarrow{\theta_Y} & \mathcal{H}_r(Y) \end{array}$$

where θ_X, θ_Y are the isomorphisms from singular to CW homology.

Hint: Go back to the definition of θ_X, θ_Y.

#4 For every map $f: S^n \to S^n$ show that $\exists$ map $g: S^n \to S^n$ such that $f \simeq g$ and g has a fixed point. $(n \geq 1)$

#5 Let S^p be the p-sphere with CW structure $S^p = e^0 \cup e^p$.

and basepoint e^0. Similarly for S^q. Show

$$S^p \times S^q / S^p \vee S^q \approx S^{p+q}$$

(See Massey p. 230 for the product of CW complexes) Note If (X, x_0), (Y, y_0) are based spaces then we can take $X \times y_0 \cup x_0 \times Y \subseteq X \times Y$ as $X \vee Y$.

#6 Show that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. (Recall we proved $\forall f : S^{2n} \to S^{2n}$, $\exists x \in S^{2n}$ with $f(x) = x$ or $f(x) = -x$. Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvalues.

#7 Construct a surjective map $S^n \to S^n$ of degree zero, $n \geq 1$. (Hint: Try S^1 first)

#8 Compute the homology of the subspace of $I \times I$ consisting of the four boundary edges plus all the points in the interior whose first coordinate is rational.